Closing Tue: 14.1,14.3(1),14.3(2)
Closing Thu: 14.4, 14.7(1)

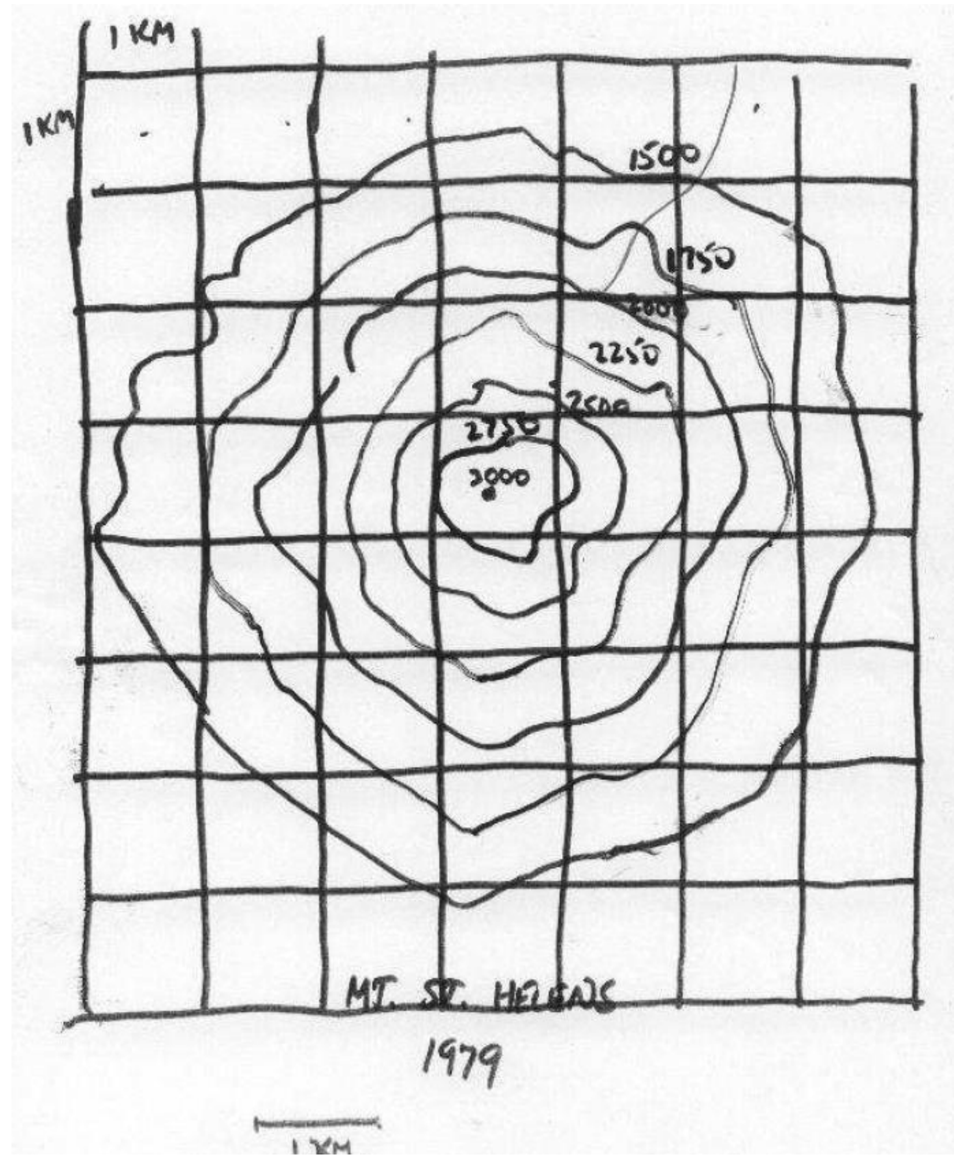
14.1/14.3 Visualizing Surfaces and Partial Derivatives

The basic tool for visualizing surfaces is **traces**.

When $z = f(x, y)$, we typically look at traces given by fixed values of z (height) first. We call these traces **level curves**, because each curve represents all the points at the same height (level).

A collection of level curves is called a **contour map** (or **elevation map**).

Contour Map (Elevation Map) of Mt. St. Helens from 1979 (before it erupted):

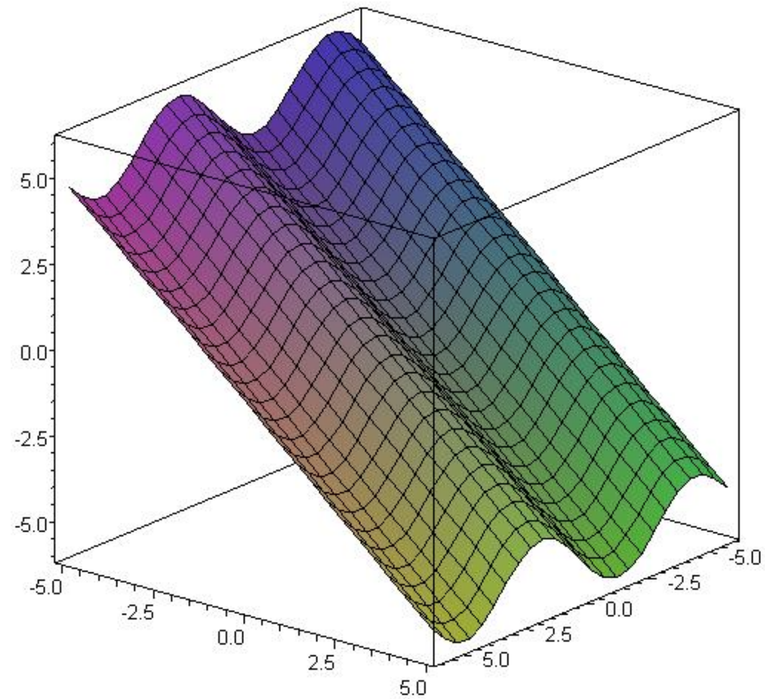


Example: Draw a contour map for

$$z = f(x, y) = y - x$$

Example: Draw a contour map for

$$z = \sin(x) - y$$

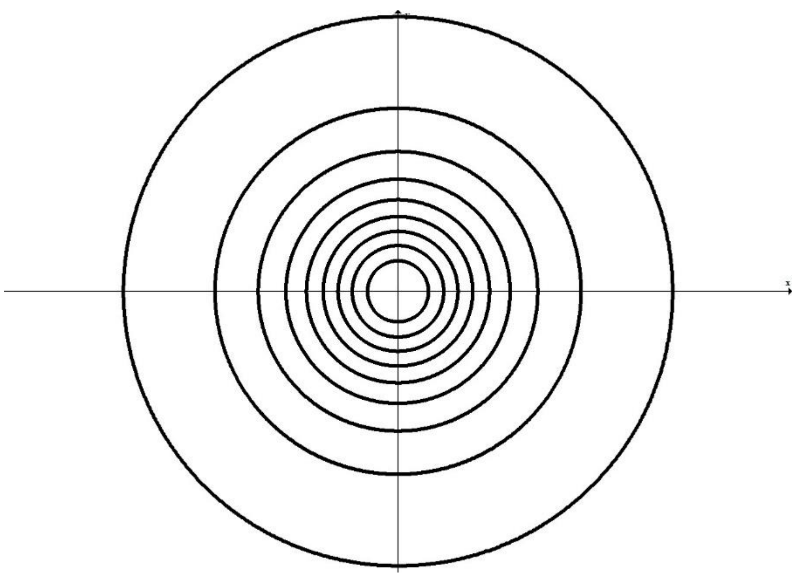


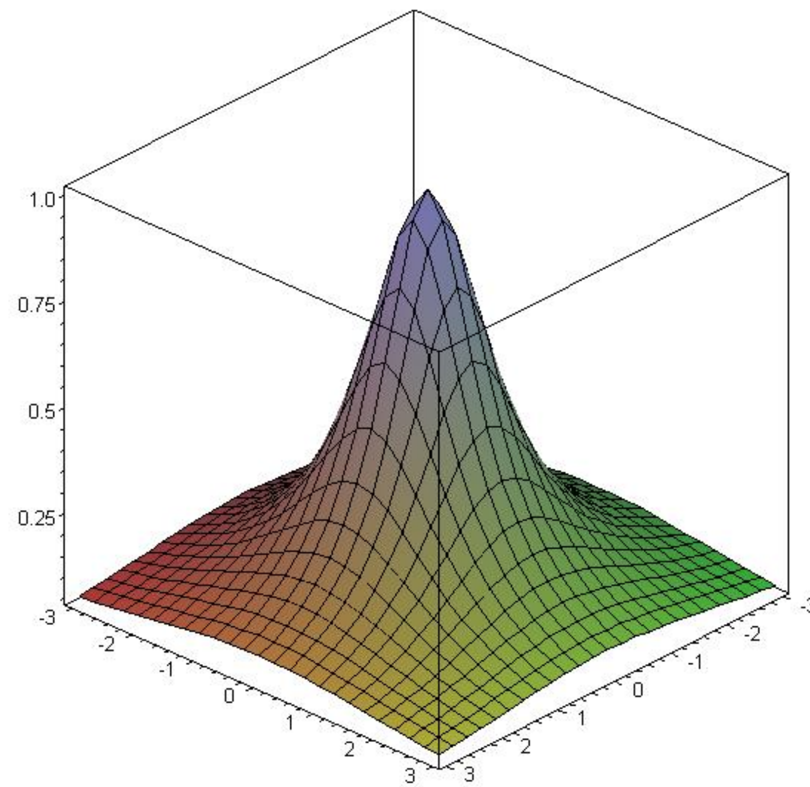
Example: Draw a contour map for

$$z = f(x, y) = \frac{1}{1 + x^2 + y^2}$$

(use z = 1/10, 2/10, … , 9/10, 10/10)

Graph of $z = f(x, y) = \frac{1}{1+x^2+y^2}$





14.3 Partial Derivatives

Goal: Get the slope in two different directions on a surface.

Recall the key definition from calculus

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Today we define:

$$\frac{\partial z}{\partial x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial z}{\partial y} = f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Motivation: Consider

$$f(x,y) = x^2y + 5x^3 + y^2$$

Find

a. $\frac{d}{dx}[f(x,2)] = \frac{d}{dx}[x^2(2) + 5x^3 + (2)^2]$

b. $\frac{d}{dx}[f(x,3)] = \frac{d}{dx}[x^2(3) + 5x^3 + (3)^2]$

c. $\frac{d}{dx}[f(x,c)] = \frac{d}{dx}[x^2(c) + 5x^3 + (c)^2]$

Example:

$$f(x, y) = x^3 y + x^5 e^{xy^2} + \ln(y)$$

Example:

$$g(x, y) = \cos(x^3 + y^4)$$

An Important Note on Variables

A variable can be treated as:

1. A constant
2. An independent variable (input)
3. A dependent variable (output),

Examples:

a) **One variable functions of *x***:

$$y = x^2$$

$$\frac{dy}{dx} =$$

b) **Related rates**:

At time *t* assume a particle is moving along the path $y = x^2$.

$$\frac{dy}{dt} =$$

c) **Implicit functions**: $x^2 + y^2 = 1$

$$\frac{dy}{dx} =$$

d) **Multivariable**:

$$z = x^2 + y^3 e^{6y} - 5xy^4$$

$$\frac{\partial z}{\partial x} =$$

$$\frac{\partial z}{\partial y} =$$

e) **Multivariable Implicit**:

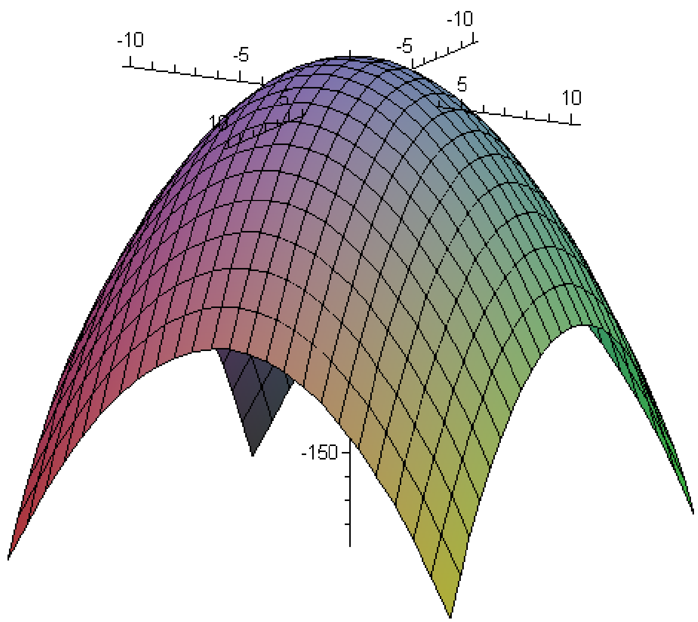
$$x^2 + y^2 - z^2 = 1$$

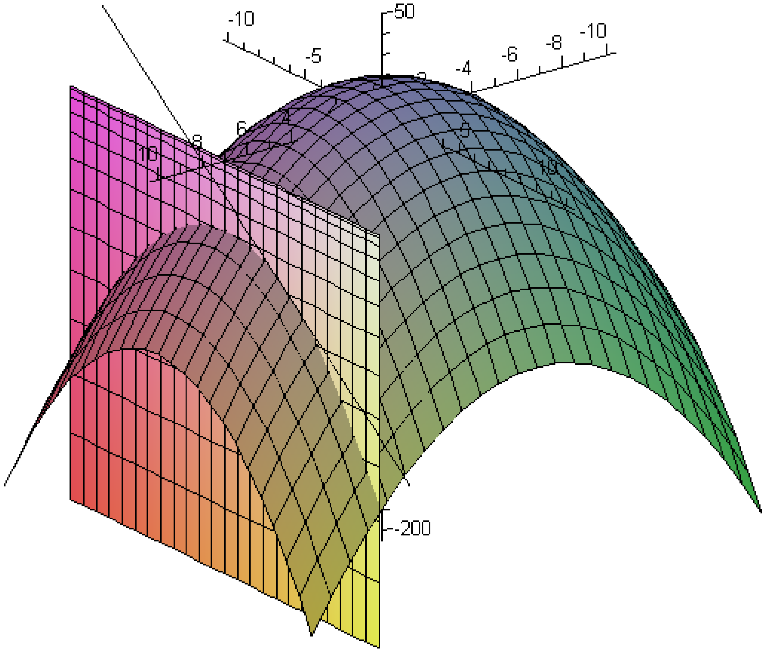
$$\frac{\partial z}{\partial x} =$$

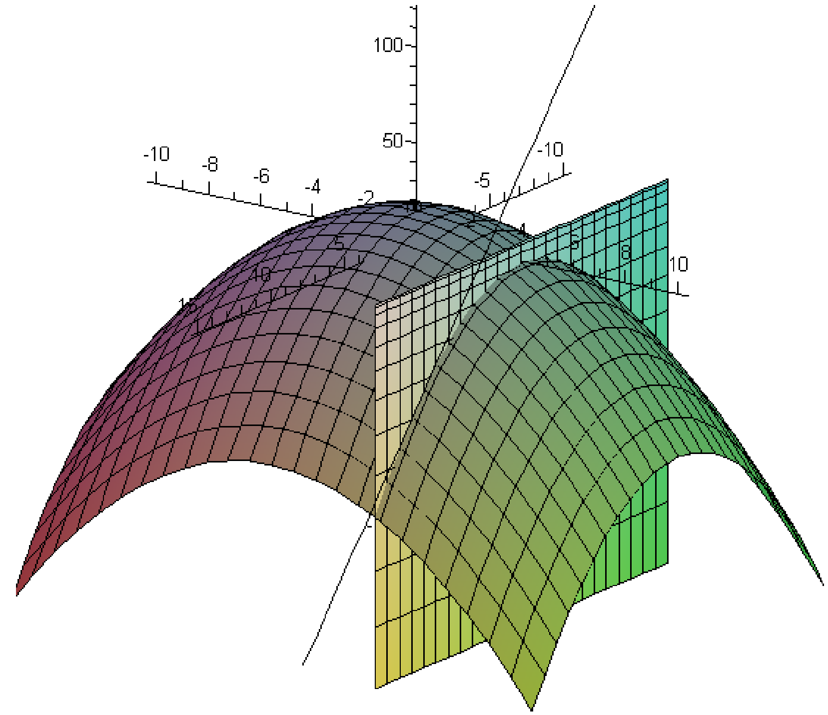
$$\frac{\partial z}{\partial y} =$$

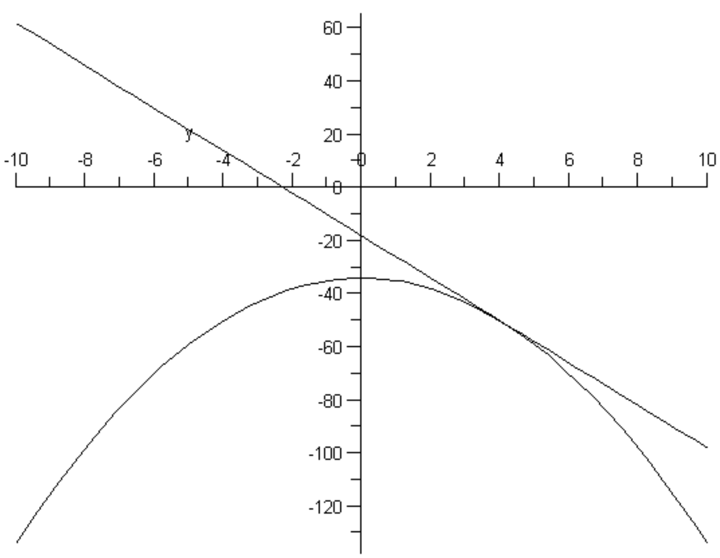
Graphical Interpretations:

Pretend you are skiing on the surface
$z = f(x, y) = 15 - x^2 - y^2$




-10
-5
-50
-4
-6
-8
-10
-200


100
50
-10
-8
-6
-4
-2
-10
-5
10


60
40
20
0
-20
-40
-60
-80
-100
-120
y
-10
-8
-6
-4
-2
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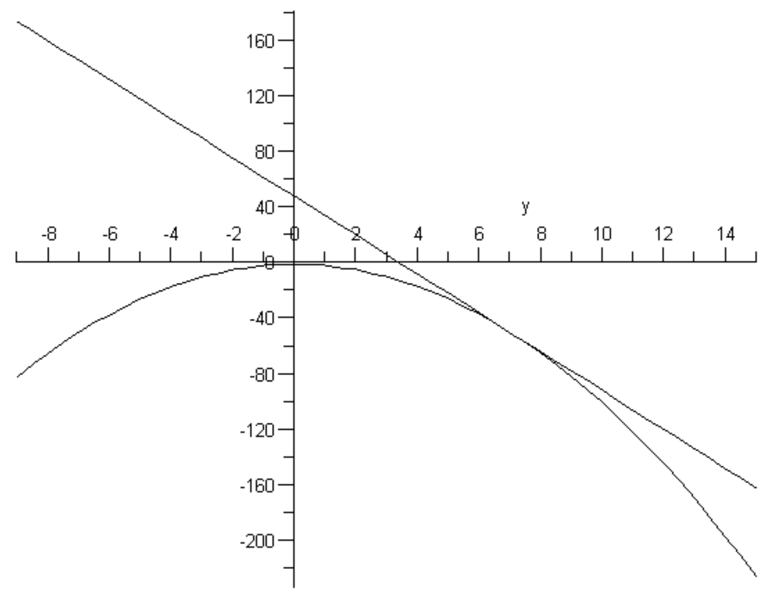
2
4
6
8
10


160
120
80
40
0
-40
-80
-120
-160
-200
y
-8
-6
-4
-2
0
2
4
6
8
10
12
14

Second Partial Derivatives

Concavity in *x*-direction:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}(x, y)$$

Concavity in *y*-direction:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy}(x, y)$$

Mixed Partials:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{xy}(x, y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{yx}(x, y)$$

Example: Find all second partials for

$z = f(x, y) = x^4 + 3x^2y^3 + y^5$